Analysis I - Semestral examination
B.Math. Hons. Ist year
First semester 2007-08
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Maximum marks 100

Q 1.
Let $A \subseteq \mathbf{R}$ be a closed set. Suppose $\{a_n\}$ is a sequence with infinitely many elements in A. If $\{a_n\}$ converges to a real number a, prove that $a \in A$.

OR

Prove that a compact subset of $\mathbf{R}$ must be closed and bounded.

Q 2.
(i) Prove that the sequence $\{\sin(n)\}$ is not convergent.
(ii) Prove that for any $0 < t < 1$, we have $\lim_{n\to\infty} nt^n = 0$.

OR

Let $\{a_n\}$ be a bounded sequence and let a be a real number such that each subsequence of $\{a_n\}$ which is convergent, converges to a. Show that $\{a_n\}$ itself must converge to a.

Q 3.
(i) For $\alpha > 0$, determine if the series $\sum_{n\geq 2} \frac{1}{n(\log\ n)^\alpha}$ converges.
(ii) Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is conditionally convergent.

OR

Let σ be a permutation of the natural numbers such that $|\sigma(n) - n| \leq 100$ for all n. If $\sum_n a_n$ is a convergent series, prove that $\sum_n a_{\sigma(n)}$ also converges.

Q 4.
Prove that a real-valued continuous function defined on a closed, bounded interval attains its infimum and supremum on it.

OR

Let $I \subset [0,1]$ be a subset such that whenever $a < b$ belong to I, the open interval $(a,b) \subset I$. Prove that I must be an interval.
P.T.O.

Q 5.
Let $f(x) = x^2 \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. Prove that $f'(0) = 0$ but that $\lim_{x\to 0} f'(x)$ does not exist.

OR

Let $g(x) = |x|^3$. Compute $g'(x), g''(x)$ by first principles and show that $g^{(3)}(0)$ does not exist.

Q 6.
Suppose $f, g : [a, b] \to \mathbf{R}$ are continuous functions which are differentiable on (a, b). If $f'(x) = g'(x)\ \forall\ x \in (a, b)$, show that $f - g$ is a constant function on $[a, b]$.

OR

Prove that the polynomial $f(x) = (x-1)(x-2)\cdots(x-100) - 1$ can have roots of multiplicity at most 2.

Q 7.
Prove that if f is a uniformly continuous on $(0, 1)$, then f can be defined at 0 and at 1 such that f is continuous at 0 and at 1.

Q 8.
Let $f : [a, b] \to \mathbf{R}$ be a continuous function which is differentiable on (a, b). Suppose that $a \leq f(x) \leq b$ for all $x \in [a, b]$ and that $f'(x) \leq \alpha < 1$ for all $x \in (a, b)$. Show that there exists a unique $c \in (a, b)$ such that $f(c) = c$.

Q 9.
Use L'Hospital's rule to evaluate :
(i) $\lim_{x\to 0+} \frac{log(\sin x)}{log(x)}$;
(ii) $\lim_{x\to 0+} \frac{\sin x}{\sqrt{x}}$;
(iii) $\lim_{x\to 1+} \frac{(x^n-1)(x^n-x)\cdots(x^n-x^{n-1})}{(x^r-1)(x^r-x)\cdots(x^r-x^{r-1})(x^{n-r}-1)(x^{n-r}-x)\cdots(x^{n-r}-x^{n-r-1})}$ for $n > r > 0$.